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An approximation to offset circle probabilities.

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AN APPROXIMATION TO OFFSET CIRCLE PROBABILITIES

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THESIS

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March 1971

An Approximation to Offset Circle Probabilities

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ABSTRACT

If a weapon system is fired at a circular target and the impact point is distributed as a bivariate normal random variable, it is not particularly difficult to determine hit probability if the expected impact point is at target center. If, for some reason, the expected impact point is offset, the problem of determining hit probabilities becomes quite complex. Herein is developed a method to approximate such a hit probability.

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I. INTRODUCTION

The probability of hitting a specified target with a given weapon system is of particular interest to both military commanders and weapon systems analysts. Determining hit probability is not always a simple matter. In particular, if the impact points of the weapon system are distributed according to some probability distribution, and the expected impact point is not at target center, the problem of determining hit probabilities becomes quite complex.

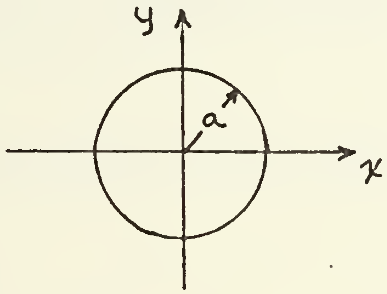
Over the past thirty years this problem has been studied in depth as the extensive bibliographies of F. Grubbs [Ref. 1], A. Eckler [Ref. 2], and W. Guenther and P. Terragno [Ref. 3] illustrate. Several approaches to solving the problem have been investigated by these and other authors in an effort to simplify the computations required to estimate or actually compute the hit probabilities.

As an example of one approach to the problem, Grubbs [Ref. 1] uses what he calls a "single, straight-forward and rather simple technique" to approximate these hit probabilities in which he employs an approximate central chi-square distribution with fractional degree of freedom, or a transformation to approximate normality. There are, of course, other methods available to approximate these hit probabilities, one of which will be discussed in the following pages.

The method which will be discussed is computationally easy, using simple mathematics in conjunction with a table of modified Bessel functions. These tables are usually available to the analyst and enable him to accurately approximate the desired hit probabilities without the use of complex computations. It should be mentioned that tables of offset circle probabilities have been compiled by Rand Corporation [Ref. 4]. These tables give hit probabilities as a function of target radius and offset distance of expected impact point, but are not as readily available as the Bessel function tables used in the following procedure. It should also be noted that routines for Bessel functions are available for most all computer systems.

II. NATURE OF THE PROBLEM

Consider a circular target with radius a. Place the origin of the coordinate system at the center of the target.



Let $f(x,y)$ be the probability density function of the impact point, (x,y), of the projectile fired at the target. Let (μ_x, μ_y) be the mean of the impact point distribution and σ_x^2 and σ_y^2 be the variance of the impact point distribution in the x and y directions respectively. Assume covariance is zero. If the distribution of impact points follows a bivariate normal distribution, then

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

Assume $\sigma_x = \sigma_y = \sigma$, and $R^2 = \mu_x^2 + \mu_y^2$. Let P_h be the probability of a hit on the target, then

$$P_h = \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2 \le a^2} \exp\left\{\frac{-1}{2\sigma^2}\left[(x-\mu_x)^2 + (y-\mu_y)^2\right]\right\} dx\, dy$$

Changing to polar coordinates yields

$$x = r\cos\theta\,, \quad y = r\sin\theta$$

and

$$P_h = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\theta \int_0^a r\,\exp\left\{-\frac{1}{2\sigma^2}\left[(r\cos\theta-\mu_x)^2+(r\sin\theta-\mu_y)^2\right]\right\} dr$$

$$= \frac{e^{-\frac{R^2}{2\sigma^2}}}{2\pi\sigma^2}\int_0^{2\pi} d\theta \int_0^a r\,\exp\left\{-\frac{1}{2\sigma^2}\left[r^2-2r(\cos\theta\mu_x+\sin\theta\mu_y\right]\right\} dr$$

$$= \frac{e^{-\frac{R^2}{2\sigma^2}}}{2\pi\sigma^2}\int_0^a r\,\exp\left(\frac{-r^2}{2\sigma^2}\right)\left[\int_0^{2\pi}\exp\left\{\frac{r}{\sigma^2}(\cos\theta\mu_x+\sin\theta\mu_y)\right\}d\theta\right]dr$$

Rotate axes so that $\mu_x = R$ and $\mu_y = 0$, then

$$P_h = \frac{e^{-\frac{R^2}{2\sigma^2}}}{2\pi\sigma^2}\int_0^a r\,\exp\left(\frac{-r^2}{2\sigma^2}\right)\left[\int_0^{2\pi}\exp\left(\frac{rR\cos\theta}{\sigma^2}\right)d\theta\right]dr$$

Looking at the term, $\int_0^{2\pi}\exp\left(\frac{rR\cos\theta}{\sigma^2}\right)d\theta$,

$$\int_0^{2\pi}\cos^n\theta\,d\theta = \cos^{n-1}\theta\sin\theta\Big|_0^{2\pi} - (n-1)\int_0^{2\pi}\cos^{n-2}\theta\sin\theta(-\sin\theta)\,d\theta$$

$$= (n-1)\int_0^{2\pi}\cos^{n-2}\theta\sin^2\theta\,d\theta$$

$$= (n-1)\int_0^{2\pi}\cos^{n-2}\theta\,d\theta - (n-1)\int_0^{2\pi}\cos^n\theta\,d\theta$$

and $\int_0^{2\pi} \cos^n\theta \, d\theta = \left(\frac{n-1}{n}\right)\int_0^{2\pi} \cos^{n-2}\theta \, d\theta$

If n is odd, then $\int_0^{2\pi} \cos^n\theta \, d\theta = 0$. If n is even, then

$\int_0^{2\pi} \cos^n\theta \, d\theta = \frac{(n-1)(n-3)(n-5)\cdots 3\cdot 1}{n(n-2)(n-4)\cdots 4\cdot 2} 2\pi$. Thus,

$$\int_0^{2\pi} \exp(x\cos\theta)\,d\theta = \sum_{n=0}^{\infty} \frac{x^n}{n!}\int_0^{2\pi} \cos^n\theta \, d\theta = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} \frac{(2m)!\, 2\pi}{2^{2m}(m!)^2}$$

$= 2\pi \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{2m} \cdot \frac{1}{(m!)^2}$. Also, for n even and equal to 2m,

$$\frac{(n-1)(n-3)(n-5)\cdots 3\cdot 1}{n(n-2)(n-4)\cdots 4\cdot 2} 2\pi = \frac{n!\, 2\pi}{[n(n-2)(n-4)\cdots 4\cdot 2]^2} = \frac{n!\, 2\pi}{\left[\left(\frac{n}{2}\right)!\right]^2 2^n}$$

$$= \frac{(2m)!\, 2\pi}{(m!)^2\, 2^{2m}}$$

The Bessel function, denoted by $J_p(x)$, is equal to

$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(n+p)!}$, where x is the argument of the Bessel function and p is the order of the Bessel function.

The modified Bessel function of order zero is denoted by $I_0(x)$, and is equal to $J_0(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{ix}{2}\right)^{2n}}{(n!)^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}$.

Thus, $\int_0^{2\pi} \exp(x\cos\theta)\,d\theta = 2\pi \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{2m} \cdot \frac{1}{(m!)^2} = 2\pi\, I_0(x)$,

so that $\int_0^{2\pi} \exp\left(\frac{rR}{\sigma^2}\cos\theta\right)d\theta = 2\pi\, I_0\left(\frac{rR}{\sigma^2}\right)$

It was shown previously that the probability of hitting the target, P_h , was equal to $\frac{e^{-\frac{R^2}{2\sigma^2}}}{2\pi\sigma^2} \int_0^a r\exp\left(\frac{-r^2}{2\sigma^2}\right)\left[\int_0^{2\pi} \exp\left(\frac{rR\cos\theta}{\sigma^2}\right)d\theta\right]dr$.

It was also shown that $\int_0^{2\pi} \exp\left(\frac{rR}{\sigma^2}\cos\theta\right)d\theta = 2\pi I_0\left(\frac{rR}{\sigma^2}\right)$.

As a result, $P_h = \frac{e^{-\frac{R^2}{2\sigma^2}}}{2\pi\sigma^2} \int_0^a r e^{-\frac{r^2}{2\sigma^2}} 2\pi I_0\left(\frac{rR}{\sigma^2}\right)dr$, or

$$P_h = e^{-\frac{R^2}{2\sigma^2}} \int_0^a \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{rR}{\sigma^2}\right)dr.$$

It is only necessary to know the ratios, $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$, to determine the probability of hitting the target,

$$P_h\left(\frac{R}{\sigma},\frac{a}{\sigma}\right) = \frac{1}{\sigma^2} e^{-\frac{R^2}{2\sigma^2}} \int_0^a r e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{rR}{\sigma^2}\right)dr.$$

The Rand Corporation has constructed tables giving these hit probabilities as a function of the two ratios, $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$, [Ref. 3].

Gilliland, [Ref. 4], shows that

$$P_h\left(\frac{R}{\sigma},\frac{a}{\sigma}\right) = e^{-\frac{R^2}{2\sigma^2}} \sum_{m=0}^{\infty} \frac{\left(\frac{R}{\sigma}\right)^{2m}}{m!\,2^m} P_m\left(\frac{a^2}{2\sigma^2}\right)$$

where $P_m\left(\frac{a^2}{2\sigma^2}\right) = 1 - e^{-\frac{a^2}{2\sigma^2}} \sum_{n=0}^{m} \left(\frac{a^2}{2\sigma^2}\right)^n \frac{1}{n!}$.

Luke, [Ref. 5], defines the function

$$J(x,y) = 1 - e^{-y}\int_0^x e^{-t} I_0\left[2(yt)^{1/2}\right]dt .$$

Recall that

$$P_h\left(\frac{R}{\sigma},\frac{a}{\sigma}\right) = \frac{1}{\sigma^2} e^{-\frac{R^2}{2\sigma^2}} \int_0^a r e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{rR}{\sigma^2}\right)dr.$$

It follows that

$$J\left(\frac{a^2}{2\sigma^2},\frac{R^2}{2\sigma^2}\right) = 1 - e^{-\frac{R^2}{2\sigma^2}} \int_0^{\frac{a^2}{2\sigma^2}} e^{-t} I_0\left[2\left(\frac{R^2}{2\sigma^2}t\right)^{1/2}\right]dt .$$

Let $t = \frac{v^2}{2\sigma^2}$, $dt = \frac{v\,dv}{\sigma^2}$. When $t = \frac{a^2}{2\sigma^2}$, $v = a$.

Thus,

$$J\left(\frac{a^2}{2\sigma^2}, \frac{R^2}{2\sigma^2}\right) = 1 - e^{-\frac{R^2}{2\sigma^2}} \int_0^a \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} I_0\left[2\left(\frac{R^2}{2\sigma^2}\cdot\frac{v^2}{2\sigma^2}\right)^{1/2}\right] dv$$

or, $$J\left(\frac{a^2}{2\sigma^2}, \frac{R^2}{2\sigma^2}\right) = 1 - \frac{1}{\sigma^2} e^{-\frac{R^2}{2\sigma^2}} \int_0^a v e^{-\frac{v^2}{2\sigma^2}} I_0\left(\frac{Rv}{\sigma^2}\right) dv.$$

As a result,

$$P_h\left(\frac{R}{\sigma}, \frac{a}{\sigma}\right) = 1 - J\left(\frac{a^2}{2\sigma^2}, \frac{R^2}{2\sigma^2}\right).$$

Luke, [Ref. 5], gives the expansion of $J(x, y)$ in a series of Bessel functions.

$$J(x,y) = e^{-(x+y)} \sum_{k=0}^{\infty} \eta^k I_k(\beta) \qquad \text{for } \eta < 1$$

$$J(x,y) = 1 - e^{-(x+y)} \sum_{k=1}^{\infty} \eta^{-k} I_k(\beta) \qquad \text{for } \eta > 1$$

where $\beta = 2(xy)^{1/2}$, and $\eta = \left(\frac{y}{x}\right)^{1/2}$.

Consider

$$J\left(\frac{a^2}{2\sigma^2}, \frac{R^2}{2\sigma^2}\right) = e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=0}^{\infty} \left(\frac{R}{a}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad \text{for } R < a$$

$$= 1 - e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=1}^{\infty} \left(\frac{a}{R}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad \text{for } R > a$$

Then

$$P_h\left(\frac{R}{\sigma}, \frac{a}{\sigma}\right) = 1 - e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=0}^{\infty} \left(\frac{R}{a}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad \text{for } R < a$$

$$= e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=1}^{\infty} \left(\frac{a}{R}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad \text{for } R > a$$

where

$R = \sqrt{\mu_x^2 + \mu_y^2}$, and $I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!(n+k)!}$.

These latter two equations yield the true hit probability for a weapon system fired at a circular target with radius a, and normally distributed impact points with expected impact point at μ_x and μ_y.

A good approximation to the true hit probability can be had by considering only the first few terms of the infinite series in the appropriate equation. That is

$$\hat{P}_h\left(\frac{R}{\sigma}, \frac{a}{\sigma}\right) = 1 - e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=0}^{n} \left(\frac{R}{a}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad R < a$$

$$= e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=1}^{n} \left(\frac{a}{R}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad R > a$$

where n is, hopefully, relatively small. If n is less than five, then the approximation would be beneficial to analysts and would be quite simple to use. If n is greater than five, the calculations required to get a reasonably accurate approximation to hit probability become prohibitive.

III. CONCLUSIONS

The accuracy of the above approximations was checked on an IBM 360 computer. The program used the FORTRAN language and two IBM subroutines, IO and INUE, which computed the modified Bessel functions. This program is available through the Operations Analysis Department of the Naval Postgraduate School in Monterey, California.

The computer program was basically an iterative process in which different values of the ratios $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$ were set and the hit probabilities computed while the number of terms from the appropriate infinite series varied from one to ten. The approximate hit probabilities thus obtained were then compared to the true hit probabilities from the Rand tables [Ref. 4].

The two aspects of primary interest were (1) how accurate are the approximating equations in the various ranges of $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$, and (2) how many terms of the infinite series in the approximating equations must be considered before a reasonable degree of accuracy is obtained. The specific values of the ratios $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$ ranged from .5 to 5.0 in the computer program. For the equation for $R<a$, the ratio $\frac{R}{\sigma}$ was initialized at the value .5 and the ratio $\frac{a}{\sigma}$ was allowed to vary between .5 and 5.0. The value for $\frac{R}{\sigma}$ was then incremented to 1.0 and the process repeated with values of $\frac{a}{\sigma}$ greater than or equal to 1.0. This procedure was continued until both $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$ were

equal to 5.0. For each value of $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$ the approximate hit probability was computed using only the first term of the infinite series initially, and then again for each additional term used, up to ten terms. In this manner it was possible to see how rapidly the approximating equations converged to the true hit probability.

For the equation for $R>a$, the same type of procedure was used, holding the ratio $\frac{a}{\sigma}$ constant while the ratio $\frac{R}{\sigma}$ varied through the appropriate values. As before, the ratio $\frac{a}{\sigma}$ was initialized at .5 and then incremented by .5 each time the ratio $\frac{R}{\sigma}$ had run its range of values and the hit probabilities computed.

Representative values of hit probabilities obtained through the approximating equations are contained in Appendix A. Figure 1 compares approximate hit probabilities with actual hit probabilities for $R<a$, and Fig. 2 does the same for $R>a$. In both cases the values shown were computed using the first five terms of the infinite series from the appropriate approximating equation.

The convergence of the approximating equation values to the true hit probabilities is shown in the two examples below. Example one shows the convergence of the approximating equation for $R<a$ as the number of terms from the infinite series increases. Values of $\frac{R}{\sigma}$ and $\frac{a}{\sigma}$ were selected to show both rapid convergence and relatively slow convergence. Example two shows convergence of the approximating equation for $R>a$. These examples along with Appendix A point out that the

approximation of hit probability for $R<a$ is optimistic while the approximation of hit probability for $R>a$ is conservative.

Example 1 ($R<a$)

$$\frac{R}{\sigma} = .5, \quad \frac{a}{\sigma} = 1.0$$

Approximation using:

one term	.2089828	
two terms	.1108707	
three terms	.1047551	
four terms	.1045011	
five terms	.1044937	True probability: .104491
six terms	.1044937	
seven terms	.1044937	
eight terms	.1044937	
nine terms	.1044937	
ten terms	.1044937	

$$\frac{R}{\sigma} = 4.5, \quad \frac{a}{\sigma} = 5.0$$

Approximation using:

one term	.9253550	
two terms	.8596848	
three terms	.8044760	
four terms	.7601165	
five terms	.7260436	True probability: .653154
six terms	.7010158	
seven terms	.6834279	
eight terms	.6715976	
nine terms	.6639764	
ten terms	.6592714	

Example 2 (R>a)

$$\frac{a}{\sigma} = .5, \quad \frac{R}{\sigma} = 1.0$$

Approximation using:

one term	.0690204
two terms	.0732899
three terms	.0734669
four terms	.0734724
five terms	.0734725
six terms	.0734725
seven terms	.0734725
eight terms	.0734725
nine terms	.0734725
ten terms	.0734725

True probability: .073473

$$\frac{a}{\sigma} = 4.5, \quad \frac{R}{\sigma} = 5.0$$

Approximation using:

one term	.0656702
two terms	.1208790
three terms	.1652384
four terms	.1993113
five terms	.2243391
six terms	.2419270
seven terms	.2537574
eight terms	.2613786
nine terms	.2660835
ten terms	.2688691

True probability: .272201

Accuracy of the approximating equations can be improved in all cases by considering more terms of the appropriate infinite series. If it is necessary to use more than five terms of the series, however, the mathematic manipulations involved become impractical to do by hand. Accuracy can also

be improved when $\frac{R}{\sigma}$ is equal to $\frac{a}{\sigma}$. From Appendix A it is apparent that when $\frac{R}{\sigma}$ is equal to $\frac{a}{\sigma}$ and both are greater than 3.0, accuracy is the poorest. To more accurately approximate the true hit probability in this case it is sufficient to compute the arithmetic mean of the values obtained from both equations ($R<a$ and $R>a$). For example, for $\frac{R}{\sigma} = \frac{a}{\sigma} = 5.0$, the true probability is .460. The equation for $R<a$ yields an approximation of .643, while the equation for $R>a$ yields an approximation of .325. The arithmetic mean of these two values is .484, which is more accurate than either of the two approximations.

IV. SUMMARY

If the impact point of a weapon system is distributed as a bivariate normal random variable, the task of computing the hit probability for a circular target when the expected impact point is somewhere other than at target center is formidable. The two approximating equations that have been developed

$$\hat{P}_h\left(\frac{R}{\sigma},\frac{a}{\sigma}\right) = 1 - e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=0}^{n} \left(\frac{R}{a}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad \text{for } R < a$$

$$= e^{-\left(\frac{a^2+R^2}{2\sigma^2}\right)} \sum_{k=1}^{n} \left(\frac{a}{R}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) \qquad \text{for } R > a$$

are relatively accurate approximations to true hit probabilities using only the first five terms of the series. Greater accuracy can be achieved by considering more terms. To have an accurate approximation to such a hit probability is beneficial to both weapon systems analysts and military commanders, because of the difficulties involved in computing actual hit probabilities. The above equations yield a good approximation to hit probabilities using simple mathematics and a table of modified Bessel functions, which is not difficult to obtain.

APPENDIX A

COMPARISON OF HIT PROBABILITIES

$\frac{a}{\sigma}$ \ $\frac{R}{\sigma}$	1.0		2.0		3.0		4.0		5.0	
	app	act	app	act	app	act	app	act	app	act
1.0	.267	.267	*	*	*	*	*	*	*	*
2.0	.731	.731	.410	.397	*	*	*	*	*	*
3.0	.956	.956	.788	.786	.498	.433	*	*	*	*
4.0	.997	.997	.966	.966	.814	.803	.578	.450	*	*
5.0	.999	.999	.998	.998	.970	.969	.835	.813	.643	.460

Figure 1. Table for R Less Than a

$\frac{R}{\sigma}$ \ $\frac{a}{\sigma}$	1.0		2.0		3.0		4.0		5.0	
	app	act	app	act	app	act	app	act	app	act
1.0	.267	.267	*	*	*	*	*	*	*	*
2.0	.082	.082	.393	.397	*	*	*	*	*	*
3.0	.011	.011	.113	.113	.399	.433	*	*	*	*
4.0	.001	.001	.015	.015	.121	.126	.366	.450	*	*
5.0	.000	.000	.001	.001	.016	.017	.120	.133	.325	.460

Figure 2. Table for R Greater Than a

These tables compare the approximation to the actual hit probabilities (app) and the actual hit probabilities (act) themselves. Only the first five terms of the infinite series described on page 11 were used for the approximations. Accuracy can be improved in all cases by considering more terms of the series.

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13. ABSTRACT

If a weapon system is fired at a circular target and the impact point is distributed as a bivariate normal random variable, it is not particularly difficult to determine hit probability if the expected impact point is at target center. If, for some reason, the expected impact point is offset, the problem of determining hit probabilities becomes quite complex. Herein is developed a method to approximate such a hit probability.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Approximation to Hit Probability						
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An approximation to offset circle probabilities.